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LEFT EXACT LOGIC

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This note gives a syntactic presentation for partial algebraic theories (see [1] and [3]). The logic, called left exact logic, is interpretable in any category with all finite limits, and it has coherent logic as a conservative extension, which implies a completeness theorem.

Left exact logic, $L^{\mathrm{e}}_{\omega\omega}$, is a fragment of coherent logic, $L^{\mathrm{g}}_{\infty\omega}$. $L^{\mathrm{e}}_{\omega\omega}$ uses the vocabulary of Horn logic with operators: terms are built up from sorted operator symbols and variables, atomic sentences are equations between same-sorted terms, sentences are conjunctions of atomic sentences. Note conjunctions in $L^{\mathrm{g}}_{\infty\omega}$ are finite, so $L^{\mathrm{e}}_{\omega\omega}$ is finitary. Sequents of $L^{\mathrm{e}}_{\omega\omega}$ are one-sided, admitting at most a single sentence as consequent.

$L^{\mathrm{e}}_{\omega\omega}$ includes the usual axioms of equality, the trivial axioms $\Sigma \Rightarrow \varphi$ whenever $\varphi \in \Sigma$, and the rules for conjunction:

$$\frac{\Sigma, \varphi \wedge \psi, \varphi \Rightarrow \chi}{\Sigma, \varphi \wedge \psi \Rightarrow \chi} \qquad \frac{\Sigma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \chi}{\Sigma, \varphi, \psi \Rightarrow \chi}$$

A theory in $L^{\mathrm{e}}_{\omega\omega}$ has a set T_A of non-logical axioms and uses the restricted cut rule

$$\frac{\Sigma, \theta, \varphi \Rightarrow \chi}{\Sigma, \theta \Rightarrow \chi} \quad \begin{array}{l}\text{when } \theta \Rightarrow \varphi \text{ is an axiom of equality or}\\ \text{an instance of a non-logical axiom}\end{array}$$

T has a set T_C of *type introduction clauses*, written

$$\text{(A)} \qquad (E(v_1, \dots, v_n) \,|\, p_1(c) = v_1 \wedge \cdots \wedge p_n(c) = v_n)$$

where $E(v_1, \dots, v_n)$ is a sentence in the indicated variables, and c is a variable not among the $v_1, \dots, v_n$, and $p_1, \dots, p_n$ are operator symbols of the language going from the sort of c to the sorts of $v_1, \dots, v_n$ respectively.

Type introduction clauses have no effect on which sentences are well formed. They are not sentences, nor sequents. They determine the theory as admitting certain rules of inference.

If T includes the clause (A), then

$$\Rightarrow E(p_1(c), \dots, p_n(c)),$$

(B)

$$p_1(c) = p_1(c') \wedge \cdots \wedge p_n(c) = p_n(c') \Rightarrow c = c'$$

are axioms of T and

$$\frac{\Sigma, E(t_1, \dots, t_n), p_1(c) = t_1 \wedge \cdots \wedge p_n(c) = t_n \Rightarrow \chi}{\Sigma, E(t_1, \dots, t_n) \Rightarrow \chi} \quad c \text{ not in conclusion}$$

is a rule of inference in T.

Intuitively, to include (A) in a theory means the sort of c is the limit, via the operators p_i, of the diagram indicated by the equations of E – the extension of E in the usual way. This intuition is justified two ways. First, we stipulate that a model of an $L^{\mathrm{e}}_{\omega\omega}$ theory with clause (A) must have the given limit as interpretation of the sort of c.

Second, it is not hard to see an $L^{\mathrm{e}}_{\omega\omega}$ theory T has an extension T^{g} in $L^{\mathrm{g}}_{\infty\omega}$ given by these two steps:

(a) Extend the language by adding ($\exists$–) and $\bigvee$ with the usual formation rules and rules of inference. Keep the original axioms (including the axioms in (B) if there are type introduction clauses).

(b) For each type introduction clause (A) in T add an axiom:

$$\text{(C)} \qquad E(v_1, \dots, v_n) \Rightarrow (\exists c)(p_1(c) = v_1 \wedge \cdots \wedge p_n(c) = v_n).$$

By the 'first main fact' in [4] any model for the coherent theory T^{g} will make the sort of c the limit of the diagram indicated by E. So any model of T in a category which has stable sups and images extends uniquely to a model of T^{g}. We will apply this for models in SET.

In fact, T^{g} is a conservative extension of T. Suppose

$$\text{(D)} \qquad \Sigma \Rightarrow \varphi$$

is a sequent of $L^{\mathrm{e}}_{\omega\omega}$ provable in T^{g} using the rules of $L^{\mathrm{g}}_{\infty\omega}$. Then $L^{\mathrm{g}}_{\infty\omega}$ has a generalized Hauptsatz saying (D) has a proof using only subsentences of sentences in Σ, φ, or the non-logical axioms of T^{g}. By construction none of these involves $\bigvee$. The only ones involving ($\exists$–) are axioms of the form (C) and it is easy to see the only results in the language of $L^{\mathrm{e}}_{\omega\omega}$ which follow from such axioms already follow in the logic of $L^{\mathrm{e}}_{\omega\omega}$ from the type introduction clause (A).

The argument applied to Horn logic in [4, p. 96] applies to $L^{\mathrm{e}}_{\omega\omega}$, proving

$$T \models \varphi \quad \text{if and only if} \quad T \overset{\mathrm{s}}{\models} \varphi$$

where $T \models \varphi$ means φ is true in every model of T in a category with finite limits, and $T \overset{\mathrm{s}}{\models} \varphi$ means φ is true in every Set-model of T.

So we have these equivalences:

$$\frac{\dfrac{\dfrac{\dfrac{T \models \varphi}{T \overset{s}{\models} \varphi}}{T^g \overset{s}{\models} \varphi}}{T^g \vdash \varphi \quad \text{in } L^g_{\infty\omega}}}{T \vdash \varphi \quad \text{in } L^e_{\omega\omega}}$$

The second step is justified by the unique extension of models in SET, the third by the completeness theorem for finitary theories in $L^g_{\infty\omega}$ (see [4, p. 162]), the fourth by the conservative extension result above.

The motivating examples of left exact theories are category theory and its partial algebraic extensions – the theory of categories with finite limits, the theory of toposes, and so on. The left exact theory of categories has two basic types: Ob and Ar, for objects and arrows respectively, and the usual function symbols: $d_0 : \mathrm{Ar} \to \mathrm{Ob}$, $d_1 : \mathrm{Ar} \to \mathrm{Ob}$ and $\mathrm{id} : \mathrm{Ob} \to \mathrm{Ar}$. Then there is a type introduction clause:

$$(d_1(f) = d_0(g) \mid p_1(c) = f \wedge p_2(c) = g)$$

with f and g of type Ar, and c of the introduced type C. C is the type of composable pairs of arrows. Finally there is a function symbol for composition $\circ : \mathrm{C} \to \mathrm{Ar}$. The theory has the obvious axioms. One could define the type of parallel pairs of arrows, and a function symbol to be axiomatized as assigning to each such pair an equalizer. Thus one can give a left exact theory of categories with all finite limits, and so on.

It may be helpful to contrast left exact logic with a similar doctrine, the universal Horn logic of Keane [2]. In a universal horn theory all types are products of the basic types and all functions are totally defined, but the axioms can require functions to satisfy given equations on equationally defined subvarieties of types. For example, there is a universal Horn theory of categories and in it composition is defined for all pairs of arrows but the axioms only apply to 'composable pairs' in the usual sense. The associative law is expressed as a conditional equation:

$$[d_1(f) = d_0(g) \wedge d_1(g) = d_0(h)] \Rightarrow f \circ (g \circ h) = (f \circ g) \circ h.$$

Universal Horn logic is the natural setting for theories such as monoids with cancellation, or rings in which all idempotents commute. It expresses, for example, the right cancellation condition on a monoid by: $xy = zy \Rightarrow x = z$.

Universal Horn logic can be interpreted in left exact logic. The right cancellation condition can be expressed this way in left exact logic: Introduce a type T

$$(xy = zy \mid p_1(t) = x \wedge p_2(t) = y \wedge p_3(t) = z)$$

so T is the type of triples x, y, z meeting the given condition. Then add an axiom: $p_1(t) = p_3(t)$. In general a universal Horn axiom $E(x_1, \dots, x_n) \Rightarrow E'(x_1, \dots, x_n)$ is expressed in left exact logic by introducing a type to be the extension of E, and asserting E' of that type.

But left exact logic can not always be interpreted in universal horn logic. Working in SET, or other categories with excluded middle, given a left exact theory one can find a universal Horn theory whose models correspond canonically to those of the left exact theory. A SET model of the left exact theory of categories can be canonically extended to a model of the universal Horn theory of categories by adding a dummy object and declaring its identity arrow to be the composite of all 'non-composable' pairs. Conversely, the interpretation above shows any model of the universal Horn theory gives a model of the left exact theory by forgetting composites of 'non-composable' pairs. But the two doctrines do not give the same morphisms of models, since all composites must be preserved by a universal Horn morphism, not only 'composable' ones. In categories without excluded middle even the canonical correspondence of categories fails.

References

[1] P.J. Freyd, Aspects of topoi, Bull. Austral. Math. Soc. 7 (1972) 1-76 and 467-480.

[2] O. Keane, Abstract Horn Theories, in F.W. Lawvere et al. eds., Model Theory and Topoi, Lecture Notes in Math. 445 (Springer, Berlin, 1975).

[3] F.W. Lawvere, Introduction to Model Theory and Topoi, Lecture Notes in Math. 445 (Springer, Berlin, 1975).

[4] M. Makkai and G.E. Reyes, First Order Categorical Logic, Lecture Notes in Math. 661 (Springer, Berlin, 1977).